

# A generalization of the category $\mathcal{O}$ of Bernstein–Gelfand–Gelfand

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## Résumé

L'étude des représentations irréductibles d'une algèbre de Lie simple définie sur le corps des nombres complexes a conduit Bernstein, Gelfand et Gelfand à introduire une catégorie qui fournit un cadre naturel pour les modules de plus haut poids. Le but de cette note est de présenter une construction d'une famille de catégories généralisant celle de Bernstein–Gelfand–Gelfand. Nous décrivons les modules simples de certaines de ces catégories. Cette classification permet de montrer que ces catégories sont semi-simples.

*Pour citer cet article :*

## Abstract

In the study of simple modules over a simple complex Lie algebra, Bernstein, Gelfand and Gelfand introduced a category of modules which provides a natural setting for highest weight modules. In this note, we define a family of categories which generalizes the BGG category. We classify the simple modules for some of these categories. As a consequence we show that these categories are semisimple.

*To cite this article:*

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## 1. Weight modules and Generalized Verma Modules

Let  $\mathfrak{g}$  denote a simple Lie algebra over  $\mathbb{C}$  and  $\mathcal{U}(\mathfrak{g})$  denote its universal enveloping algebra. Let  $\mathfrak{h}$  be a fixed Cartan subalgebra and denote by  $\mathcal{R}$  the corresponding set of roots. For  $\alpha \in \mathcal{R}$  we denote by  $H_\alpha \in \mathfrak{h}$  the corresponding co-root and by  $\mathfrak{g}^\alpha$  the root space for the root  $\alpha$ . We will denote by  $\Delta$  a set of simple roots. We write  $\mathcal{R}^+$  for the corresponding set of positive roots. For a subset  $\theta \subset \Delta$ , we denote by  $\langle \theta \rangle$  the set of all roots which are linear combination of elements of  $\theta$  and set  $\langle \theta \rangle^\pm = \langle \theta \rangle \cap \mathcal{R}^\pm$ . We consider the following subalgebras of  $\mathfrak{g}$  :

$$\mathfrak{l}_\theta = \mathfrak{h} \oplus \sum_{\alpha \in \langle \theta \rangle} \mathfrak{g}^\alpha, \quad \mathfrak{n}_\theta^\pm = \sum_{\alpha \in \mathcal{R}^\pm - \langle \theta \rangle^\pm} \mathfrak{g}^\alpha.$$

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The subalgebra  $\mathfrak{p}_\theta = \mathfrak{l}_\theta \oplus \mathfrak{n}_\theta^+$  is called the standard parabolic subalgebra associated to  $\theta$  and  $\mathfrak{l}_\theta$  is the standard Levi subalgebra. The latter is a reductive algebra. Its semisimple part is denoted  $\mathfrak{l}'_\theta$ . If  $\theta = \emptyset$ , then  $\mathfrak{l}_\theta = \mathfrak{h}$  and we simply write  $\mathfrak{n}^+$  instead of  $\mathfrak{n}_\theta^+$ .

We denote by  $\text{Mod}(\mathfrak{g})$  the category of all  $\mathfrak{g}$ -modules. We will investigate some full subcategories of  $\text{Mod}(\mathfrak{g})$  for which we will describe the simple modules. A module  $M$  is a *weight module* if it is finitely generated, and  $\mathfrak{h}$ -diagonalizable in the sense that

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda, \quad M_\lambda = \{m \in M : H \cdot m = \lambda(H)m\},$$

with weight spaces  $M_\lambda$  of finite dimension. We will denote by  $\mathcal{M}(\mathfrak{g}, \mathfrak{h})$  the full subcategory of  $\text{Mod}(\mathfrak{g})$  consisting of all weight modules. This category has been studied by several authors (e.g. [7], [8]). This category also appears as a particular case of generalised Harish-Chandra modules (see [14] for a definition).

The Bernstein–Gelfand–Gelfand category  $\mathcal{O}$  is the full subcategory of  $\mathcal{M}(\mathfrak{g}, \mathfrak{h})$  whose objects are  $\mathfrak{n}^+$ -finite (a module  $M$  is  $\mathfrak{n}^+$ -finite if for all  $m \in M$  the set  $\mathcal{U}(\mathfrak{n}^+)m$  is a finite dimensional vector space). The category  $\mathcal{O}$  was introduced by Bernstein, Gelfand and Gelfand (see [2]). There they describe the simple modules and the structure of the category itself, namely they study the projective modules in  $\mathcal{O}$  and give a certain correspondence for multiplicities (the so called BGG-correspondence). For a review of these results we refer the reader to [10]. In order to discuss the simple modules in  $\mathcal{M}(\mathfrak{g}, \mathfrak{h})$  we need to recall some well known facts about generalised Verma modules.

A  $\mathfrak{l}_\theta$ -module can be made into a  $\mathfrak{p}_\theta$ -module by letting  $\mathfrak{n}_\theta^+$  act trivially. For such a module  $N$  we define the *generalised Verma module* (GVM)  $V(\theta, N)$  by  $V(\theta, N) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p}_\theta)} N$ . For any  $\mathfrak{g}$ -module  $M$  we define the  $\mathfrak{l}_\theta$ -module  $M^{\mathfrak{n}_\theta^+} := \{m \in M \mid \mathfrak{n}_\theta^+ m = 0\}$ . Let us recall the following classical facts about GVMs:

**Proposition 1.1** (i) *If  $N$  is a simple  $\mathfrak{l}_\theta$ -module, the module  $V(\theta, N)$  admits a unique simple quotient, denoted by  $L(\theta, N)$ . The module  $L(\theta, N)$  is called the simple  $\mathfrak{g}$ -module induced from  $(\mathfrak{l}_\theta, N)$ .*

(ii) *If  $M$  is a simple  $\mathfrak{g}$ -module such that  $M^{\mathfrak{n}_\theta^+} \neq \{0\}$ , then  $M \cong L(\theta, M^{\mathfrak{n}_\theta^+})$ .*

We refer to [7, proposition 3.8] for a proof. We refer to [13] for a more detailed discussion of GVM's.

To give the classification of simple weight modules, we need one more ingredient: the so-called *cuspidal* modules.

## 2. Cuspidal modules

Let  $M$  be a weight module. A root  $\alpha \in \mathcal{R}$  is said to be *locally nilpotent* (with respect to  $M$ ) if  $X_\alpha \in \mathfrak{g}^\alpha$  acts by a locally nilpotent operator on the whole module  $M$ . It is said to be *cuspidal* if  $X_\alpha$  acts injectively on the whole module  $M$ . We denote by  $\mathcal{R}^N(M)$  the set of locally nilpotent roots and by  $\mathcal{R}^I(M)$  the set of cuspidal roots. We shall simply denote  $\mathcal{R}^N$  and  $\mathcal{R}^I$  when the module  $M$  is clear from the context.

It is known that for a simple weight module  $\mathcal{R} = \mathcal{R}^N \sqcup \mathcal{R}^I$  (see [7, lemma 2.3]). A weight module is called *cuspidal* if  $\mathcal{R} = \mathcal{R}^I$ . Set  $\mathcal{R}_s^N = \{\alpha \in \mathcal{R}^N : -\alpha \in \mathcal{R}^N\}$ ,  $\mathcal{R}_a^N = \mathcal{R}^N - \mathcal{R}_s^N$ .

We define  $\mathcal{R}_s^I$  and  $\mathcal{R}_a^I$  the same way. Recall the following theorem:

**Theorem 2.1 (Fernando [7, theorem 4.18], Futorny [8])** *Let  $M$  be a simple weight module. Then  $M$  is induced from a cuspidal simple module of some Levi subalgebra of  $\mathfrak{g}$ .*

More precisely there is a set of simple roots  $\Delta$  and a subset  $\theta \subset \Delta$  such that  $\langle \theta \rangle = \mathcal{R}_s^I$  and  $\mathcal{R}^+ - \langle \theta \rangle^+ = \mathcal{R}^N$ . Then  $M^{\mathfrak{n}_\theta^+}$  is a simple cuspidal module for  $\mathfrak{l}_\theta$  and  $M \cong L(\theta, M^{\mathfrak{n}_\theta^+})$ .

The theorem of Fernando reduces the classification of simple weight  $\mathfrak{g}$ -modules to the classification of simple cuspidal weight modules for reductive Lie algebras. By standard arguments this reduces to the classification of simple cuspidal modules for simple Lie algebras. A first step towards this classification is given by the following theorem:

**Theorem 2.2 (Fernando [7, theorem 5.2])** *Let  $\mathfrak{g}$  be a simple Lie algebra. If  $M$  is a simple cuspidal  $\mathfrak{g}$ -module, then  $\mathfrak{g}$  is of type  $A$  or  $C$*

The classification of simple cuspidal modules was then completed in two steps. In the first step Britten and Lemire classified all simple cuspidal modules of degree 1 (see [5]) where  $\deg(M) = \sup_{\lambda \in \mathfrak{h}^*} \{\dim(M_\lambda)\}$ .

Britten and Lemire, and later Benkart, Britten and Lemire, have classified all simple modules of degree 1 when  $\mathfrak{g}$  is of type  $A$  or  $C$  (see [1]). These modules will play a key role in our theorem 3.2 below.

Later Mathieu gave the full classification of simple cuspidal modules of finite degree greater than 1 by introducing the notion of a coherent family (see [12]).

### 3. The category $\mathcal{O}_{S,\theta}$

Now we define a family of full subcategories of  $\mathcal{M}(\mathfrak{g}, \mathfrak{h})$ . Fix a set of simple roots  $\Delta$ . Fix two subsets  $S$  and  $\theta$  of  $\Delta$  such that  $\theta \subset S$ . The category  $\mathcal{O}_{S,\theta}$  is the full subcategory of  $\mathcal{M}(\mathfrak{g}, \mathfrak{h})$  whose objects are the modules satisfying the following conditions :

- (O1) As a  $\mathfrak{l}_\theta$ -module,  $M$  is a direct sum of simple highest weight modules.
- (O2) For  $\alpha \in \langle S - \theta \rangle$ , the element  $X_\alpha \in \mathfrak{g}^\alpha$  is cuspidal for  $M$ .
- (O3) The module  $M$  is  $\mathfrak{n}_S^+$ -finite.

Notice that in the case  $S = \theta = \emptyset$ , we recover the definition of the category  $\mathcal{O}$ . Other generalisations of category  $\mathcal{O}$  can be recovered for particular choices of  $(\theta, S)$  (see [15], [13]). We sometimes write  $\mathcal{O}_{S,\theta}(\mathfrak{g})$  to emphasize the Lie algebra. Now we list some easy properties of  $\mathcal{O}_{S,\theta}$  which are generalisations of analogous properties of  $\mathcal{O}$ .

**Proposition 3.1** *Let  $M, N \in \mathcal{O}_{S,\theta}$ .*

- (i) *Then  $M \oplus N$  is in  $\mathcal{O}_{S,\theta}$ . Every submodule and every quotient of  $M$  is again in  $\mathcal{O}_{S,\theta}$ .*
- (ii) *The category  $\mathcal{O}_{S,\theta}$  is abelian, and every module in  $\mathcal{O}_{S,\theta}$  is noetherian and artinian.*
- (iii) *The category  $\mathcal{O}_{S,\theta}$  is  $\mathcal{Z}(\mathfrak{g})$ -finite (where  $\mathcal{Z}(\mathfrak{g})$  is the center of  $\mathcal{U}(\mathfrak{g})$ ).*
- (iv) *The simple modules in  $\mathcal{O}_{S,\theta}$  are modules of the form  $L(S, N)$  where  $N$  is a simple module in  $\mathcal{O}_{S,\theta}(\mathfrak{l}_S)$ .*

Being artinian and noetherian, every module  $M$  in  $\mathcal{O}_{S,\theta}$  admits a finite Jordan–Hölder series whose quotients are of the form  $L(S, N)$  (see [11]). This allows us to define the multiplicity of  $L(S, N)$  in any Jordan–Hölder series of  $M$ . From proposition 3.1(iv), the description of the simple modules in  $\mathcal{O}_{S,\theta}(\mathfrak{g})$  first requires to consider the case where  $S = \Delta$ .

In the sequel we assume  $S = \Delta$ . We shall only consider a simple Lie algebra  $\mathfrak{g}$  and restrict ourselves to the case  $\theta \neq \emptyset$  and  $\theta \neq \Delta$ . We label the simple roots as in [3]. First, from theorem 2.1, every simple module in  $\mathcal{O}_{\Delta,\theta}$  is of the form  $L(\Delta - \theta, C)$  for some simple cuspidal  $\mathfrak{l}_{\Delta-\theta}$ -module  $C$ . Such a module is defined by a simple cuspidal  $\mathfrak{l}'_{\Delta-\theta}$ -module  $C'$  and a scalar action of the center  $\mathfrak{z}$  of  $\mathfrak{l}_{\Delta-\theta}$ . We have to find those  $C$  for which  $L(\Delta - \theta, C)$  satisfies condition (O1) of the definition of the category  $\mathcal{O}_{\Delta,\theta}$ .

**Theorem 3.2** *Assume  $(\mathfrak{g}, \Delta - \theta)$  is not one of the following:  $(B_n, \{e_1\})$ ,  $(D_n, \{e_1\})$ ,  $(D_n, \{e_{n-1}\})$ ,  $(D_n, \{e_n\})$ ,  $(E_6, \{e_1\})$ ,  $(E_6, \{e_6\})$ ,  $(E_7, \{e_7\})$ . Then we have*

- (i) *If  $L(\Delta - \theta, C)$  is a simple module in  $\mathcal{O}_{\Delta,\theta}(\mathfrak{g})$ , then  $\mathfrak{g}$  is of type  $A_n$  or is of type  $C_n$ ,  $\Delta - \theta$  is a connected subset of  $\Delta$  and  $C$  is a simple cuspidal module of degree 1.*
- (ii) *If  $\mathfrak{g} = \mathfrak{sl}_{n+1}$  and  $\Delta - \theta$  is any connected subset of  $\Delta$  other than  $\{e_1\}$  and  $\{e_n\}$ , then the simple modules in  $\mathcal{O}_{\Delta,\theta}(\mathfrak{g})$  are modules of degree 1. Conversely, any infinite dimensional simple  $\mathfrak{sl}_n$ -module of degree 1 is a simple object in some category  $\mathcal{O}_{\Delta,\theta}(\mathfrak{sl}_n)$ .*
- (iii) *If  $\mathfrak{g} = \mathfrak{sp}_n$ , then there are simple modules in  $\mathcal{O}_{\Delta,\theta}(\mathfrak{g})$  if and only if  $\Delta - \theta \supset \{e_n\}$ . In this case, all the simple modules in  $\mathcal{O}_{\Delta,\theta}(\mathfrak{g})$  are modules of degree 1. Conversely, any infinite dimensional simple  $\mathfrak{sp}_n$ -module of degree 1 is a simple object in some category  $\mathcal{O}_{\Delta,\theta}(\mathfrak{sp}_n)$ .*

From now on, we assume that  $\mathfrak{g}$  is of type  $A$  or  $C$  and that  $\theta$  is neither empty nor equal to  $\Delta$ . When  $\mathfrak{g} = A_n$  we also assume that  $\Delta - \theta$  is different from  $\{e_1\}$  and  $\{e_n\}$ . When the category  $\mathcal{O}_{\Delta, \theta}$  is non empty, we want to describe its structure. First we note the following fact: if  $F$  is a finite dimensional  $\mathfrak{g}$ -module of dimension greater than 1 and  $M \in \mathcal{O}_{\Delta, \theta}$  is simple, then  $F \otimes M$  does not belong to the category  $\mathcal{O}_{\Delta, \theta}$  in general. The proof of this fact requires some results of Britten and Lemire [6, section 3]. Therefore the category  $\mathcal{O}_{\Delta, \theta}$  has a structure very different from the case of the category  $\mathcal{O}$ . In fact, we prove the following

**Theorem 3.3** *Let  $M$  and  $N$  be two simple modules in  $\mathcal{O}_{\Delta, \theta}$ . Then  $\text{Ext}_{\mathcal{O}_{\Delta, \theta}}^1(M, N) = 0$ .*

As a corollary we get:

**Corollary 3.4** *The category  $\mathcal{O}_{\Delta, \theta}$  is semisimple.*

Remark that the result holds trivially when  $\theta = \Delta$  (this is in fact a part of the definition of the category). Note that when  $\theta = \emptyset$  and  $\mathfrak{g}$  is of type  $C$ , Britten, Khomenko, Lemire, Mazorchuk have proved the semisimplicity of  $\mathcal{O}_{\Delta, \theta}$  in [4]. The result does not hold when  $\theta = \emptyset$  and  $\mathfrak{g}$  is of type  $A$ . In [9], Grantcharov and Serganova have described the indecomposable modules in this case. The proof of theorem 3.3 uses the example 3.3 of [9].

Detailed proofs will be published elsewhere.

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